Generalized Belief Function, Plausibility Function, and Dempster’s Combinational Rule to Fuzzy Sets

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Uncertainty always exists in nature and real systems. It is known that probability has been used traditionally in modeling uncertainty. Since a belief function was proposed as an another type of measuring uncertainty, Dempster-Shafer theory (DST) has been widely studied and applied in diverse areas. Because of the advent of computer technology, the representation of human knowledge can be processed by a computer in complex systems. The analysis of fuzzy data becomes increasingly important. Up to date, there are several generalizations of DST to fuzzy sets proposed in the literature. In this article, we propose another generalization of belief function, plausibility function, and Dempster’s combinational rule to fuzzy sets. We then make the comparisons of the proposed extension with some existing generalizations and show its effectiveness. © 2003 Wiley Periodicals, Inc.

1. INTRODUCTION

In 1967, Dempster1 proposed upper and lower probabilities using a multivalued mapping. Let \((\Omega, G, P)\) be a probability space and let \(T\) be a multivalued mapping from \(\Omega\) to \(S\), which assigns a subset \(T(\omega) \subseteq S\) to every \(\omega \in \Omega\); i.e., \(T\) is a set-valued function from \(\Omega\) to the power set \(P_S\) of \(S\). For any \(A \in P_S\), define

\[
A_\# = \{\omega \in \Omega | T(\omega) \subseteq A, T(\omega) \neq \emptyset\}
\]

and

\[
A^* = \{\omega \in \Omega | T(\omega) \cap A \neq \emptyset\}
\]

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In particular, \( S^* = S_* = \Omega \). Let \( H \) be a class of subsets \( A \) of \( S \) so that \( A_* \) and \( A^* \) belong to \( G \), Dempster\(^1\) defined the lower probability \( P_* \) and upper probability \( P^* \) of \( A \) in \( H \) as
\[
P_*(A) = P(A_*) / P(S^*) \quad \text{and} \quad P^*(A) = P(A^*) / P(S^*)
\]
where \( P_*(A) \) and \( P^*(A) \) are defined only if \( P(S^*) \neq 0 \). It can be shown that \( P_*(A) = 1 - P^*(A^c) \), \( P_*(A) + P_*(A^c) \leq 1 \), \( P^*(A) + P^*(A^c) \geq 1 \), and \( P^*_*(A) \leq P^*(A) \). Note that if \( T \) is a (single-valued) function, then \( T \) becomes a random variable and \( P_*(A) = P_*(A) \). If \( T \) is a multivalued mapping, then an outcome \( \omega \) may be mapped to more than one value. This is similar to one case having more than two interpretations or points of view. Dempster\(^2\) then used it to infer hypotheses and a generalized Bayesian inference.

Suppose that \( S \) is a finite set. Shafer\(^3–5\) defined a set function \( m : P_S \rightarrow [0, 1] \) with the conditions (1) \( m(\emptyset) = 0 \) and (2) \( \sum_{A \in P_S} m(A) = 1 \). The set function \( m \) is called a basic probability assignment (BPA). An element \( A \) in \( P_S \) with \( m(A) \neq 0 \) is called a focal element. Note that if \( m \) is defined as \( m(B) = \sum_{T(\omega) = B} P(\{\omega\}) \) with \( m(\emptyset) = 0 \) for any \( B \in P_S \) and the mapping \( T \) is a function, then \( m \) becomes an induced probability measure. Shafer called \( P_*(A) \) a belief (Bel) function and \( P^*(A) \) a plausibility (Pl) function and showed that for any \( A \in P_S \),
\[
\begin{align*}
\text{Bel}(A) &= P_*(A) = \sum_{B \subseteq A} m(B) \\
\text{Pl}(A) &= P^*(A) = \sum_{B \cap A \neq \emptyset} m(B) \\
\text{Bel}(A) &= 1 - \text{Pl}(A^c)
\end{align*}
\]

In addition, it should be mentioned that Zadeh\(^16\) proposed a possibility theory on the basis of fuzzy sets. Eventually, possibility becomes one type of plausibility.

All of these (nonadditivity) measures may be regarded as measures for the information content of an evidence, and the theory usually is called the Dempster-Shafer theory (DST; see Refs. 7–9). Since the advent of computer technology, the representation of human knowledge can be processed by a computer in complex systems. The analysis of fuzzy-valued data becomes increasingly important (see Ref. 10). Yang et al.\(^11–13\) had treated fuzzy-valued data in clustering, possibility, and fuzzy regression analysis. Zadeh\(^14\) first generalized the DST to fuzzy sets. Afterward, more generalizations were made by several different researchers (see Refs. 15–20). In this article, an advanced generalization of Bel, Pl, and Dempster’s combinational rule to fuzzy sets is proposed. In Section 2, we propose the generalized Bel and Pl functions in the DST by extending Bel and Pl functions to fuzzy sets. We then make the comparisons with some other existing generalizations. Section 3 presents an extension of Dempster’s combination rule and conclusions are made in Section 4.
2. A GENERALIZATION OF Bel AND Pl FUNCTIONS TO FUZZY SETS

After Zadeh\textsuperscript{14} proposed information granularity and extended Shafer’s Bel function to fuzzy sets, Ishizuka et al.,\textsuperscript{15} Ogawa et al.,\textsuperscript{17} and Yager\textsuperscript{19} made similar extensions according to a different defined measure $I(\bar{A} \subseteq \bar{B})$ of fuzzy inclusion for which

$$\text{Bel}(\bar{B}) = \sum_{\bar{A}} I(\bar{A} \subseteq \bar{B}) m(\bar{A})$$

They defined the measure of fuzzy inclusion as follows:

Ishizuka et al.\textsuperscript{15}:

$$I(\bar{A} \subseteq \bar{B}) = \frac{\min_x \{1, 1 + (\mu_B(x) - \mu_A(x))\}}{\max_x \mu_A(x)}$$

Ogawa et al.\textsuperscript{17}:

$$I_O(\bar{A} \subseteq \bar{B}) = \frac{\sum_x \min \{\mu_A(x), \mu_B(x)\}}{\sum_x \mu_B(x)}$$

Yager\textsuperscript{19}:

$$I_Y(\bar{A} \subseteq \bar{B}) = \min_x \{\max \mu_A(x), \mu_B(x)\}$$

Afterward, Yen\textsuperscript{20} formulated linear programming problems for computing the Bel and Pl functions of fuzzy sets. He showed that the optimal solutions of linear programming problems are the minimum and maximum probability masses that can be allocated to a fuzzy set $\bar{B}$ from a (crisp) set $A$ as follows:

$$m_*(\bar{B} : A) = m(A) \times \inf_{x \in A} \mu_B(x)$$

$$m^*(\bar{B} : A) = m(A) \times \sup_{x \in A} \mu_B(x)$$

To deal with a fuzzy focal element $\bar{A}$, Yen\textsuperscript{20} used the resolution form $\bar{A} = \bigcup_a \alpha A_a$ and defined the probability assignment as follow:

$$m(\bar{A}_a) = (\alpha_i - \alpha_{i-1}) \times m(\bar{A})$$

where $\alpha_0 = 0, \alpha_n = 1, \alpha_{i-1} < \alpha_i$, $i = 1, \ldots, n$. He defined the probability assignment that a fuzzy focal $\bar{A}$ contributes to a fuzzy set $\bar{B}$ as

$$m_*(\bar{B} : \bar{A}) = \sum_{\alpha} m_*(\bar{B} : \bar{A}_\alpha)$$

$$m^*(\bar{B} : \bar{A}) = \sum_{\alpha} m^*(\bar{B} : \bar{A}_\alpha)$$
Finally, the formulas for computing the Bel and Pl functions of fuzzy sets are obtained:

\[
\text{Bel}(\tilde{B}) = \sum_{\tilde{A}} m_{\tilde{A}}(\tilde{B} : \tilde{A}) = \sum_{\tilde{A}} m(\tilde{A}) \sum_{\alpha_i} (\alpha_i - \alpha_{i-1}) \inf_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}}(x)
\]

\[
\text{Pl}(\tilde{B}) = \sum_{\tilde{A}} m^*(\tilde{B} : \tilde{A}) = \sum_{\tilde{A}} m(\tilde{A}) \sum_{\alpha_i} (\alpha_i - \alpha_{i-1}) \sup_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}}(x)
\]

(11)

Note that if the conditional factor \(C(\tilde{B} : \tilde{A}) = \sum_{\alpha_i}(\alpha_i - \alpha_{i-1})\inf_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}}(x)\), of Equation 11 is replaced by \(C(\tilde{B} : \tilde{A}) = \min_{\{x|\mu_{\tilde{A}}(x) > 0\}} \mu_{\tilde{B}}(x)\), then the Bel function of fuzzy sets becomes

\[
\text{Bel}(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A}) \min_{\{x|\mu_{\tilde{A}}(x) > 0\}} \mu_{\tilde{B}}(x)
\]

(12)

which is defined by Smets.\(^{18}\)

Now, we extend the Bel and Pl functions to fuzzy sets along with Yen’s approach.\(^{20}\) Let \(\theta_\alpha = \{x|\mu_{\tilde{A}}(x) = \alpha\}, \alpha \in [0, 1]\). Define

\[
m_{\tilde{B}}(\tilde{B} : \tilde{A}_\alpha) = m'(\tilde{A}_\alpha) \times \inf_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}}(x)
\]

(13)

\[
m'(\tilde{A}_\alpha) = m(\tilde{A}) \times \frac{|\theta_\alpha|}{|\tilde{A}|}
\]

(14)

where \(|\tilde{A}| = \sum_{x} \mu_{\tilde{A}}(x)\) and \(|\theta_\alpha| = \sum_{x \in \tilde{A}_\alpha} \mu_{\tilde{A}}(x)\) are the cardinalities of \(\tilde{A}\) and \(\tilde{\theta}_\alpha\) respectively. The Bel function of a fuzzy set \(\tilde{B}\) is formulated as

\[
\text{Bel}(\tilde{B}) = \sum_{\tilde{A}} m_{\tilde{A}}(\tilde{B} : \tilde{A})
\]

\[
= \sum_{\tilde{A}} \sum_{\alpha} m_{\tilde{A}}(\tilde{B} : \tilde{A}_\alpha) \quad \text{(according to a resolution form} \tilde{A} = \bigcup_{\alpha} \alpha\tilde{A}_\alpha) \]

\[
= \sum_{\tilde{A}} \sum_{\alpha} m'(\tilde{A}_\alpha) \times \inf_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}}(x)
\]

\[
= \sum_{\tilde{A}} m(\tilde{A}) \sum_{\alpha} \frac{|\theta_\alpha|}{|\tilde{A}|} \times \inf_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}}(x)
\]

(15)

Similarly, the plausibility function of a fuzzy set \(\tilde{B}\) is given as

\[
\text{Pl}(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A}) \sum_{\alpha} \frac{|\theta_\alpha|}{|\tilde{A}|} \times \sup_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}}(x)
\]

(16)

The contribution of a fuzzy focal element \(\tilde{A}\) is decomposed to the contribution of each crisp set \(\tilde{A}_\alpha\) under level \(\alpha\). The contribution proportion of \(\tilde{A}_\alpha\) to overall contribution of \(\tilde{A}\) is considered as the membership proportion of elements with membership \(\alpha\) to overall memberships. This approach can catch more information.
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to the change of fuzzy focal elements according to its generalization structure. We now discuss some properties of our generalized Bel and Pl measures.

**Properties.**

1. If $\tilde{B}_1 \subset \tilde{B}_2$, then $\text{Bel}(\tilde{B}_1) \leq \text{Bel}(\tilde{B}_2)$
2. $\text{Bel}(\tilde{B}) = 1 - \text{Pl}(\tilde{B}^c)$
3. $\text{Bel}(\tilde{B}) + \text{Bel}(\tilde{B}^c) \leq 1$ and $\text{Pl}(\tilde{B}) + \text{Pl}(\tilde{B}^c) \geq 1$

where $\tilde{B}_1 \subset \tilde{B}_2$ if $\mu_{\tilde{B}_1}(x) \leq \mu_{\tilde{B}_2}(x)$ for all $x$ and $\mu_{\tilde{B}}(x) = 1 - \mu_{\tilde{B}^c}(x)$.

**Proof.**

1. If $\tilde{B}_1 \subset \tilde{B}_2$ then $\inf_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}_1}(x) = \inf_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}_2}(x)$ for all $\alpha$, this implies that $\text{Bel}(\tilde{B}_1) \leq \text{Bel}(\tilde{B}_2)$.
2. $1 - \text{Pl}(\tilde{B}^c) = 1 - \sum_{\alpha} m(\tilde{A}) \sum_{\alpha} \|\theta_{\alpha}/\tilde{A}\| \times \sup_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}}(x) = \sum_{\alpha} m(\tilde{A}) \sum_{\alpha} \|\theta_{\alpha}/\tilde{A}\| \times (1 - \sup_{x \in \tilde{A}_\alpha} \mu_{\tilde{B}}(x)) = \text{Bel}(\tilde{B})$
3. Since $0 \leq \text{Bel}(\tilde{B}) \leq 1$, $0 \leq \text{Pl}(\tilde{B}) \leq 1$, and $\text{Bel}(\tilde{B}) \leq \text{Pl}(\tilde{B})$ this implies $-1 \leq \text{Bel}(\tilde{B}) - \text{Pl}(\tilde{B}) \leq 0$ and hence $\text{Bel}(\tilde{B}) + \text{Bel}(\tilde{B}^c) = \text{Bel}(\tilde{B}) + 1 - \text{Pl}(\tilde{B}) \leq 1$. Similarly, $\text{Pl}(\tilde{B}) + \text{Pl}(\tilde{B}^c) \geq 1$.

Here, the cardinality of $|\tilde{A}|$ is considered finite. To compare the proposed Bel function with the other Bel functions, we use an example similar to Yen’s to illustrate.

**Example 1.** Let $S = \{1, 2, \ldots, 10\}$. Let $\tilde{A}$ and $\tilde{C}$ be fuzzy sets in $\tilde{S}$ with

$\tilde{A} = \{0.25/1, 0.5/2, 0.75/3, 1/4, 1/5, 0.75/6, 0.5/7, 0.25/8\}$

$\tilde{C} = \{0.5/5, 1/6, 0.8/7, 0.4/8\}$

where each member of the list is in the form of $\mu_{\tilde{A}}(x_i)/x_i$. Let $\tilde{B}$ be a fuzzy set in $\tilde{P}_S$ with

$\tilde{B} = \{0.5/2, 1/3, 1/4, 1/5, 0.9/6, 0.6/7, 0.3/8\}$

The decomposition of the fuzzy focal $\tilde{A}$ consists of four nonfuzzy focal elements:

$\tilde{A}_{0.25} = \{1, 2, \ldots, 8\}$ with mass $m(\tilde{A}_{0.25}) = 0.1 \times m(\tilde{A})$

$\tilde{A}_{0.5} = \{2, 3, \ldots, 7\}$ with mass $0.2 \times m(\tilde{A})$

$\tilde{A}_{0.75} = \{3, 4, 5, 6\}$ with mass $0.3 \times m(\tilde{A})$

$\tilde{A}_{1.0} = \{4, 5\}$ with mass $0.4 \times m(\tilde{A})$

and the decomposition of the fuzzy focal $\tilde{C}$ consists of four nonfuzzy focal elements:
\( C_{0.4} = \{5, 6, 7, 8\} \) with mass \( 0.148 \times m(\tilde{C}) \),
\( C_{0.5} = \{5, 6, 7\} \) with mass \( 0.186 \times m(\tilde{C}) \),
\( C_{0.8} = \{6, 7\} \) with mass \( 0.296 \times m(\tilde{C}) \),
\( C_{1.0} = \{6\} \) with mass \( 0.370 \times m(\tilde{C}) \).

Then,
\[
m_{\#}(\mathcal{B} : \tilde{A}) = \sum_{\tilde{A}} m(\tilde{A}) \sum_{\alpha} \left| \frac{\theta_{\alpha}}{|\tilde{A}|} \right| \times \inf_{x \in \tilde{A}} \mu_{\tilde{A}}(x)
\]
\[
= m(\tilde{A})(0.1 \times 0 + 0.2 \times 0.5 + 0.3 \times 0.9 + 0.4 \times 1)
\]
\[
= 0.77 \times m(\tilde{A})
\]
\[
m_{\#}(\mathcal{B} : \tilde{C}) = m(\tilde{C})(0.148 \times 0.3 + 0.186 \times 0.6 + 0.296 \times 0.6 + 0.370 \times 0.9) = 0.666 \times m(\tilde{C})
\]
Thus, we obtain
\[
\text{Bel}(\tilde{B}) = 0.77 \times m(\tilde{A}) + 0.666 \times m(\tilde{C})
\]

Similarly, we have
\[
\text{Pl}(\tilde{B}) = m(\tilde{A}) + 0.9334m(\tilde{C})
\]

The degrees of Bel in the fuzzy set \( \tilde{B} \) computed using the other methods are listed as follows:

Ishizuka et al.\textsuperscript{15}: Bel(\( \tilde{B} \)) = 0.75m(\( \tilde{A} \)) + 0.8m(\( \tilde{C} \))

Ogawa et al.\textsuperscript{17}: Bel(\( \tilde{B} \)) = 0.8962m(\( \tilde{A} \)) + 0.434m(\( \tilde{C} \))

Yager\textsuperscript{19}: Bel(\( \tilde{B} \)) = 0.5m(\( \tilde{A} \)) + 0.6m(\( \tilde{C} \))

Yen\textsuperscript{20}: Bel(\( \tilde{B} \)) = 0.6m(\( \tilde{A} \)) + 0.54m(\( \tilde{C} \))

We compare how these results are changed in response to a change in a fuzzy focal element, such as Yen’s work,\textsuperscript{20} by setting
\[
\tilde{A}’ = \{0.166/1, 0.5/2, 0.833/3, 1/4, 1/5, 0.75/6, 0.5/7, 0.25/8\}
\]
\[
\tilde{A}” = \{0.25/1, 0.75/2, 1/3, 1/4, 1/5, 0.75/6, 0.5/7, 0.25/8\}
\]
\[
\tilde{A}”” = \{0/1, 0.5/2, 0.75/3, 1/4, 1/5, 0.75/6, 0.5/7, 0.25/8\}
\]

Similar to Yen,\textsuperscript{20} Table I lists the portion of each modified focal mass that contributes to \( \tilde{B} \)’s Bel measure and Table II shows how Bel(\( \tilde{B} \)) computed by different methods changes as the focal element \( \tilde{A} \) changes in three ways. According to the results of Table II, the comparison indicates that our method is similar to Yen’s method\textsuperscript{20} in that it is more responsive to any change in a focal element’s
Example 2. In this example, we compare our Bel function with Smets'\textsuperscript{18} and Yen’s\textsuperscript{20} Bel functions. Continuing with Example 1, recall that

\[
\tilde{B} = \{0.5/2, 1/3, 1/4, 1/5, 0.9/6, 0.6/7, 0.3/8\}
\]

Let \(\tilde{D}\) be a fuzzy set in \(\tilde{S}\) with

\[
\tilde{D} = \{0.75/2, 0.9/3, 1/4, 1/5, 0.5/6, 0.25/7, 0.1/8\}
\]

Now, compare how the results from Smets’,\textsuperscript{18} Yen’s,\textsuperscript{20} and our Bel functions are changed in response to a change in a fuzzy focal element. Let \(D_1 \sim D_6\) be fuzzy sets constituting six changes in \(\tilde{D}\) with

\[
\begin{align*}
D_1 &= \{1/4, 1/5\} \\
D_2 &= \{0.9/3, 1/4, 1/5\} \\
D_3 &= \{0.75/2, 0.9/3, 1/4, 1/5\} \\
D_4 &= \{0.75/2, 0.9/3, 1/4, 1/5, 0.5/6\} \\
D_5 &= \{0.75/2, 0.9/3, 1/4, 1/5, 0.5/6, 0.25/7\} \\
D_6 &= \{0.75/2, 0.9/3, 1/4, 1/5, 0.5/6, 0.25/7, 0.1/8\} = \tilde{D}
\end{align*}
\]

As in Example 1, we can analyze the impact on the Bel functions by comparing the contributions of the focal element \(\tilde{D} = \tilde{D}_6\) and its variations \(\tilde{D}_1 = \tilde{D}_5\) to the degree of Bel in \(\tilde{B}\). Table III lists the portion of each modified focal’s mass that contributes to \(\tilde{B}\)’s Bel measure [i.e., the ratio \(m_D(\tilde{B} : \tilde{D})/m(\tilde{D})\) for each method].

### Table I. Contribution to Bel(\(\tilde{B}\)) from the focal element \(\tilde{A}\) and its variations.

<table>
<thead>
<tr>
<th>Focal element</th>
<th>Yager\textsuperscript{19}</th>
<th>Ishizuka et al.\textsuperscript{15}</th>
<th>Ogawa et al.\textsuperscript{17}</th>
<th>Yen\textsuperscript{20}</th>
<th>Ours</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{A})</td>
<td>0.5</td>
<td>0.75</td>
<td>0.8962</td>
<td>0.6</td>
<td>0.77</td>
</tr>
<tr>
<td>(\tilde{A}')</td>
<td>0.5</td>
<td>0.834</td>
<td>0.9119</td>
<td>0.6252</td>
<td>0.8466</td>
</tr>
<tr>
<td>(\tilde{A}'')</td>
<td>0.5</td>
<td>0.75</td>
<td>0.9434</td>
<td>0.5</td>
<td>0.727</td>
</tr>
<tr>
<td>(\tilde{A}'')</td>
<td>0.5</td>
<td>1</td>
<td>0.8962</td>
<td>0.675</td>
<td>0.8263</td>
</tr>
</tbody>
</table>

### Table II. Changes to Bel(\(\tilde{B}\)) caused by changes in the focal element \(\tilde{A}\).

<table>
<thead>
<tr>
<th>Changes of focal element of (\tilde{A})</th>
<th>Yager\textsuperscript{19}</th>
<th>Ishizuka et al.\textsuperscript{15}</th>
<th>Ogawa et al.\textsuperscript{17}</th>
<th>Yen\textsuperscript{20}</th>
<th>Ours</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{A} \rightarrow \tilde{A}')</td>
<td>(U)</td>
<td>(I)</td>
<td>(I)</td>
<td>(I)</td>
<td>(I)</td>
</tr>
<tr>
<td>(\tilde{A} \rightarrow \tilde{A}'')</td>
<td>(U)</td>
<td>(U)</td>
<td>(I)</td>
<td>(D)</td>
<td>(D)</td>
</tr>
<tr>
<td>(\tilde{A} \rightarrow \tilde{A}'')</td>
<td>(U)</td>
<td>(I)</td>
<td>(U)</td>
<td>(I)</td>
<td>(I)</td>
</tr>
</tbody>
</table>

\(U, I,\) and \(D\) denote unchanged, increased, and decreased, respectively.
Hence, the comparisons in Table III indicate that our generalization of the Bel measure applied to fuzzy-valued data actually give a decreasing contribution to \( \text{Bel}(\tilde{B}) \) from the changes \( \tilde{D}_1 \sim \tilde{D}_6 \) in the focal element in increasing fuzziness and also responsive to every change in the focal element. However, those of Smets\(^{18} \) and Yen\(^{20} \) are not always responsive to a change in the focal element.

In general, Yen’s Bel function\(^{20} \) could not measure the changing information to a focal element’s change unless it results in a change in the “critical point,” a point in which its membership value is the minimal value on its decomposition of the focal element. For example the focal element \( \tilde{D}_2 \) can be decomposed into two crisp sets \( \tilde{D}_{2,1.0} = \{4, 5\} \) and \( \tilde{D}_{2,0.9} = \{3, 4, 5\} \). These two decomposed crisp sets have the critical point with regard to fuzzy set \( \tilde{B} \) with any of 3, 4, and 5 [because \( \mu_{\tilde{B}}(3) = \mu_{\tilde{B}}(4) = \mu_{\tilde{B}}(5) = 1.0 \)]. Thus, the contributions of \( \tilde{D}_1 \) and \( \tilde{D}_2 \) to \( \tilde{B} \) are both the same because they do not result in any change in a critical point in its minimal membership value. The focal elements \( \tilde{D}_3, \tilde{D}_4, \) and \( \tilde{D}_5 \) have the same critical point with 2 (its membership value = 0.5) for decomposed crisp sets \( \tilde{D}_{3,0.75}, \tilde{D}_{4,0.75}, \tilde{D}_{4,0.5}, \tilde{D}_{5,0.75}, \) and \( \tilde{D}_{5,0.25} \). Thus, the contributions of \( \tilde{D}_3, \tilde{D}_4, \) and \( \tilde{D}_5 \) to \( \tilde{B} \) do not make any change (all with the contribution \( 0.25 \times 1 + 0.75 \times 0.5 = 0.625 \)). Until \( \tilde{D}_6 \), the critical point for \( \tilde{D}_{6,0.1} \) changes to 8 with a membership value of 0.3. The contribution of \( \tilde{D}_6 \) to \( \tilde{B} \) changes to \( 0.25 \times 1 + 0.65 \times 0.5 + 0.1 \times 0.3 = 0.605 \).

On the other hand, Smets’ Bel function\(^{18} \) could not measure the changing information to the change in a focal element either unless it results in a change in the critical point, a point in which its membership value is the minimal value over the focal element. Note that Smets\(^{18} \) did not consider the decomposition of a focal element. For example, the critical point of \( \tilde{D}_2 \) is any of 3, 4, and 5 with the membership value of 1.0. The critical point of \( \tilde{D}_3, \tilde{D}_4, \) and \( \tilde{D}_5 \) is 2 with the membership value of 0.5 and that of \( \tilde{D}_6 \) is 8 with the membership value of 0.3. Thus, the contributions of \( \tilde{D}_1 \sim \tilde{D}_6 \) are 1.0, 1.0, 0.5, 0.5, 0.5, and 0.3.

It is seen that our method for computing Bel measures is on the basis of weighting on the cardinality of the decomposition of the focal element. Our method actually could catch the changing information to the change of focal elements. In summary, the foregoing comparisons indicate that our method is more effective and responsive to any change in fuzziness and membership value of a focal element than the other methods.

The foregoing extension of Bel function to fuzzy sets is considered only for \(|\tilde{A}| \) finite and \( \mu_{\tilde{A}}(x) \) discrete types. If \(|\tilde{A}| = \infty \), then we define

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**Table III.** Contribution to Bel(\( \tilde{B} \)) from focal element and its variations.

<table>
<thead>
<tr>
<th>Focal element of</th>
<th>Smets(^{18} )</th>
<th>Yen(^{20} )</th>
<th>Ours</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{D}_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \tilde{D}_2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \tilde{D}_3 )</td>
<td>0.5</td>
<td>0.625</td>
<td>0.8973</td>
</tr>
<tr>
<td>( \tilde{D}_4 )</td>
<td>0.5</td>
<td>0.625</td>
<td>0.8494</td>
</tr>
<tr>
<td>( \tilde{D}_5 )</td>
<td>0.5</td>
<td>0.625</td>
<td>0.8295</td>
</tr>
<tr>
<td>( \tilde{D}_6 )</td>
<td>0.3</td>
<td>0.605</td>
<td>0.8178</td>
</tr>
</tbody>
</table>
This is similar to Smets’ method.18

3. AN EXTENSION TO DEMPSTER’S COMBINATION RULE

Let Bel1 and Bel2 be two Bel functions of crisp sets over the same frame of discernment according to two independent evidential sources. If m1 and m2 are BPAs of Bel1 and Bel2, respectively, then the combined BPA for a crisp set C is computed by Dempster’s rule of combination as

\[
m_1 \oplus m_2(C) = \frac{\sum_{A \cap B = C} m_1(A)m_2(B)}{1 - \sum_{A \cap B = \emptyset} m_1(A)m_2(B)}
\]  

Ishizuka et al.15 extended Dempster’s rule of combination to fuzzy sets by taking into account the degree J(\(\tilde{A}, \tilde{B}\)) of fuzzy intersection of two fuzzy sets as

\[
m_1 \oplus m_2(\tilde{C}) = \frac{\sum_{A \cap B = \tilde{C}} J(\tilde{A}, \tilde{B})m_1(A)m_2(B)}{1 - \sum_{A \cap B = \emptyset} (1 - J(\tilde{A}, \tilde{B}))m_1(A)m_2(B)}
\]

where

\[
J(\tilde{A}, \tilde{B}) = \frac{\max_x \mu_{\tilde{A} \cap \tilde{B}}(x)}{\min[\max_x \mu_{\tilde{A}}(x), \max_x \mu_{\tilde{B}}(x)]}
\]

Yen20 extended Dempster’s rule of combination to fuzzy sets by using two operators with a cross-product operation and a normalization process as follows:

1. Cross-product

\[
m'(\tilde{C}) = m_1 \otimes m_2(\tilde{C}) = \sum_{A \cap B = \tilde{C}} m_1(A)m_2(B)
\]

2. Normalization

\[
N[m'](\tilde{D}) = \frac{\sum_{\tilde{C} \subseteq \tilde{D}} \mu_{\tilde{C}}(x)m'(\tilde{C})}{1 - \sum_{\tilde{C}} (1 - \max_x \mu_{\tilde{C}}(x))m'(\tilde{C})}
\]

where \(\tilde{C}\) is the normalization of \(\tilde{C}\). The normalization of fuzzy focal elements is needed for Yen’s approach.20 Suppose that \(\tilde{A} = \{0.5/1, 0.8/2\}\), the decomposition of this fuzzy focal element for computing the Bel function are \(\tilde{A}_{0.5} = \{1, 2\}\) with mass 0.5\(m(\tilde{A})\), \(\tilde{A}_{0.8} = \{2\}\) with mass 0.3\(m(\tilde{A})\), and \(\tilde{A}_{1.0} = \emptyset\) with mass 0.2\(m(\tilde{A})\). The empty set \(\emptyset\) will give contributions and the focal elements are normalized to avoid this problem. In the special case that there are only two fuzzy BPAs to be combined and all fuzzy focal elements are normalized, the combined BPA of a subnormal fuzzy set can be
We propose another extension of Dempster’s rule of combination to fuzzy sets by constructing a weighted variable $W(\tilde{C}, \tilde{A})$, which expresses the weight of contribution to the fuzzy set $\tilde{C}$ from a focal element $\tilde{A}$. It is defined as

$$W(\tilde{C}, \tilde{A}) = \frac{|\tilde{C}|}{|\tilde{A}|}$$

(21)

which will be equivalent to Ishizuka’s generalization. 15

We propose another extension of Dempster’s rule of combination to fuzzy sets by constructing a weighted variable $W(\tilde{C}, \tilde{A})$, which expresses the weight of contribution to the fuzzy set $\tilde{C}$ from a focal element $\tilde{A}$. It is defined as

$$m_1 \oplus m_2(\tilde{C}) = \frac{\sum_{\tilde{A} \cap \tilde{B} \cap \tilde{C}} \max_x \mu_{\tilde{A} \cap \tilde{B}}(x)m_1(\tilde{A})m_2(\tilde{B})}{1 - \sum_{\tilde{A} \cap \tilde{B}} (1 - \max_x \mu_{\tilde{A} \cap \tilde{B}}(x))m_1(\tilde{A})m_2(\tilde{B})}$$

which will be equivalent to Ishizuka’s generalization. 15

Example 3. Let $S = \{2, 3, 4, 5, 6\}$ and let all focal elements be as follows:

$\tilde{A}_1 = \{0.75/2, 0.5/3, 0.75/4, 1/5\}$

$\tilde{A}_2 = \{0.5/3, 1/4, 0.5/5\}$

$\tilde{A}_3 = \{0.25/2, 1/3, 0.75/4\}$

$\tilde{A}_4 = \{0.5/5, 1/6\}$

$\tilde{A}_5 = \{0.25/2, 1/4, 0.75/6\}$

Assume that there are two experts assigning the two BPAs $m_1$ and $m_2$ listed in Table IV. The three different methods are used to compute the combined fuzzy BPA $m_1 \oplus m_2$ with the results listed in Table V. To make comparisons of different methods, how the the results are changed in response to a change in the fuzzy focal element $\tilde{A}_1$ can be seen. We change $\tilde{A}_1$ to $\tilde{A}_1'$, $\tilde{A}_1''$, and $\tilde{A}_1'''$ so that the combined fuzzy focal elements $\tilde{C}_1$ to $\tilde{C}_8$ are not changed where $\tilde{A}_1'$, $\tilde{A}_1''$, and $\tilde{A}_1'''$ are listed as

$\tilde{A}_1' = \{1/2, 0.5/3, 0.75/4, 1/5\}$

$\tilde{A}_1'' = \{0.5/2, 0.5/3, 0.75/4, 1/5\}$

$\tilde{A}_1''' = \{0.75/2, 0.5/3, 0.75/4, 0.5/5\}$
The results in Table VI show how the combined BPA calculated by different methods changes as the fuzzy focal element $A_1$ changes with regard to $A_1^I$, $A_1^U$, and $A_1^D$.

The comparisons in Table VI indicate that our generalized Dempster’s rule of combination applied to fuzzy sets could catch more changing informations than the others. Generally, Ishizuka’s methods could not measure the changing information to changes in a focal element unless it results in a change in the degree $J(A_1, B_1)$ of fuzzy intersection. Yen’s method could not measure the changing information to the change in a focal element either because it directly extends Dempster’s rule of combination and then uses the normalization process.

It is mentioned that Isizuka’s, Yen’s, and our extensions of Dempster’s rule of combination are not associative, i.e.,

| Combined focal element | Ishizula $^{15}$ | Yen $^{20}$ | Ours
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$ = $A_1 \cap A_2$</td>
<td>0.2416</td>
<td>0.2416</td>
<td>0.2851</td>
</tr>
<tr>
<td>$C_2$ = $A_1 \cap A_3$</td>
<td>0.1812</td>
<td>0.1812</td>
<td>0.1571</td>
</tr>
<tr>
<td>$C_3$ = $A_1 \cap A_4$</td>
<td>0.0403</td>
<td>0.0403</td>
<td>0.0078</td>
</tr>
<tr>
<td>$C_4$ = $A_2 \cap A_3$</td>
<td>0.1208</td>
<td>0.1208</td>
<td>0.0465</td>
</tr>
<tr>
<td>$C_5$ = $A_2 \cap A_4$</td>
<td>0.1074</td>
<td>0.1074</td>
<td>0.1862</td>
</tr>
<tr>
<td>$C_6$ = $A_3 \cap A_4$</td>
<td>0.0604</td>
<td>0.0604</td>
<td>0.0545</td>
</tr>
<tr>
<td>$C_7$ = $A_3 \cap A_5$</td>
<td>0.0134</td>
<td>0.0134</td>
<td>0.0039</td>
</tr>
<tr>
<td>$C_8$ = $A_4 \cap A_5$</td>
<td>0.0537</td>
<td>0.0537</td>
<td>0.0233</td>
</tr>
</tbody>
</table>

$\hat{C}_1 = \{0.5/3, 0.75/4, 0.5/5\}, \hat{C}_2 = \{0.25/2, 0.5/3, 0.75/4\}, \hat{C}_3^I = \hat{C}_3^U = \{0.5/5\}, \hat{C}_4^I = \hat{C}_4^U = \{0.25/2, 0.75/4\}, \hat{C}_5 = \{0.5/3, 1/4, 0.5/5\}, \hat{C}_6^I = \hat{C}_6^U = \{0.5/3, 0.75/4\}, \hat{C}_7 = \{1/4\},$ and $\hat{C}_8 = \{0.25/2, 1/3, 0.75/4\}.$
It is known that association is important for any evidence combination rule. For solving this problem, our generalization is divided into two steps that are analogous to Yen’s process. Hence, we first calculate fuzzy BPAs by applying Dempster’s rule of combination directly to fuzzy focal elements and then continue to combine all fuzzy BPAs without normalization. Finally, the normalization process using a weighted variable is applied at the end so that the summation of fuzzy BPAs is equal to one. The two steps in the combination rule are formulated as follows:

**Step 1.**

\[
m_1 \oplus \cdots \oplus m_n(\tilde{C}) = \sum_{\tilde{A}_1 \cap \cdots \cap \tilde{A}_n = \tilde{C}} m_1(\tilde{A}_1) \cdots m_n(\tilde{A}_n) = m_{1 \cdots n}(\tilde{C})
\]

**Step 2.**

\[
M[m_1 \oplus \cdots \oplus m_n](\tilde{B}) = \frac{\sum_{\tilde{C} \subseteq \tilde{B}} W(\tilde{B} : \tilde{A}_1) \cdots W(\tilde{B} : \tilde{A}_n)m_{1 \cdots n}(\tilde{C})}{1 - \sum_{\tilde{A}_1, \ldots, \tilde{A}_n} (1 - W(\tilde{B} : \tilde{A}_1) \cdots W(\tilde{B} : \tilde{A}_n))m_{1 \cdots n}(\tilde{C})}
\]

### 4. CONCLUSIONS

In this article, the generalized DST to fuzzy sets was considered. We especially focused on generalized Bel, Pl functions and Dempster’s combination rule to fuzzy sets. We extended the Bel and Pl functions for processing fuzzy data and showed that our generalization could catch more changing information to the change of fuzzy focal elements than Yen’s and others. We also gave good properties of the generalized Bel and Pl functions. In the generalization of Dempster’s combination rule, the degree of fuzzy intersection of two fuzzy sets was divided into two weighted variables. The proposed combination rule is more effective to the change of fuzzy focal elements than Ishizuka’s and Yen’s generalization.

**Acknowledgment**

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**References**